Average lifetime and geometric properties for superlong chaotic transients in a hybrid optical bistable system

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(Received 1 November 1995)

We study the geometric structure of superlong chaotic transients which have been observed in a hybrid optical bistable system, and demonstrate that the supertransients are due to the uncertainty exponent arbitrarily close to zero. It is reported that the average lifetime of chaotic transients increases with an increase of the Lyapunov dimension of the chaotic saddle. [S1063-651X(96)12107-3]

PACS number(s): 05.45.+b, 42.65.Sf, 42.65.Pc

A phenomenon very common in dynamical systems is that they seem to behave chaotically during some transient period, but at last fall onto a periodic attractor. This phenomenon is said to be chaotically transient. It is observed in many dynamical systems such as the Hénon map [1,2], the Lorenz model [3], and the Mackey-Glass delay equation [4,5]. For those low-dimensional systems [1–3], superlong chaotic transients (referred to as ''supertransients'') can occur in the event of a boundary crisis where a chaotic attractor is suddenly destroyed and is converted into a nonattracting chaotic saddle as a system parameter passes through a critical value. In this sense, supertransients occur only in an arbitrarily small parameter interval in the vicinity of the critical value, but for high-dimensional systems, superlong chaotic transients are common. It was observed in numerical experiments by Cruchfield and Kaneko that spatially extended systems generically exhibit long chaotic transients [6]. Hastings and Higgins [7] pointed out the existence of complex transient dynamics in simple discrete-time ecological models for a species with alternating reproduction and dispersal. More recently, Lai and Winslow [8] studied the geometric properties of the chaotic saddle responsible for supertransients in spatiotemporal chaotic systems, and demonstrated that supertransients are due to a nonattracting saddle whose stable manifold measures have fractal dimensions that are arbitrarily close to the phase space dimension, and that the average lifetime of the chaotic transient induced by the chaotic saddle is arbitrarily large. In experimental research, Vallée and Delisle [9] and Giacomelli *et al.* [10] showed the existence of long-lived transients in a dynamical system with a delayed feedback. Jánosi, Flepp and Tél [11] carried out a time series analysis of transient chaos in a NMR laser experiment, and determined characteristics such as dimension, Lyapunov exponent, and correlation function.

In physical systems exhibiting transient chaos there exists in phase space a nonattracting chaotic set and a chaotic saddle, that in the dissipative case coexists with an attractor [12–14]. Because the uncertainty exponent (which will be discussed in the following) is very close to zero, the initial condition randomly chosen in phase space is very close to the stable manifold of the chaotic saddle. The trajector first is attracted to the chaotic saddle along the stable manifold, and wanders in the vicinity of the chaotic saddle for a long time before settling into the final attractor.

Let Γ denote a nonattracting chaotic saddle, and Λ a nonchaotic attractor in phase space. All initial conditions, except for a set of measure zero, eventually asymptote to Λ. Trajectories starting from random initial conditions typically wander chaotically near the chaotic saddle Γ for a finite time before setting into Λ. The length of the chaotic transient depends on its initial condition. Let $M(t)$ denote the number of trajectories staying still inside Γ after time t, and take M_0 initial conditions so large that $M(t) \gg 1$. As t becomes large, one observes, in general, an exponential decay in the number of survivors, that is, one finds asymptotically [12] that

$$\frac{M(t)}{M_0} \propto e^{-t/\tau}, \tag{1}$$

where τ is the average lifetime of the chaotic transient. The dimension of the stable manifold of the chaotic saddle is $N-1+d_S$ [8,13], where N is the phase-space dimension, and d_S is the fractal dimension of the set of intersecting points of a one-dimensional line with the stable manifold of chaotic set. d_S can be computed using the uncertainty algorithm introduced by McDonald *et al.* [15] to calculate the dimension of fractal basin boundaries for dynamical systems with multiple attractors. For a given perturbation ε, a fraction of uncertain initial condition $f(\varepsilon)$ can be computed by randomly choosing many initial conditions. For fractal sets, $f(\varepsilon)$ decreases with decreasing ε, typically scaling with ε as

$$f(\varepsilon) \propto \varepsilon^{\alpha}, \tag{2}$$

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where α is the uncertainty exponent. The fraction dimension $d_S=1-\alpha$ [15,16]. Reference [9] demonstrates numerically that d_S is arbitrarily close to 1.

Trajectories starting from points of a chaotic saddle (or of its stable manifold) never leave the saddles, and exhibit chaotic motion forever. It is, however, completely unlikely to hit such a point by random choice, since the saddle is a set of zero measure (a fractal) and is globally not attractive. Trajectories starting close to the saddle can stay in its neighborhood for a long time, and show chaotic properties. In this case, if the largest Lyapunov exponent is calculated during the transient, it is positive. Sooner or later trajectones escape the neighborhood, and tend toward the periodic attractor. Physically, let us note that in experimental situations one never has infinitely long time intervals. In fact, what is needed for an experimental observation of chaos is a well defined separation of time scales.

In this paper, we first investigate the geometric structure of the chaotic saddle for supertransients in a hybrid optical bistable system, which can be described by a delayed differential equation [17]

$$\frac{dx(t)}{dt}=-x(t)+A\ \sin^2[x(t-t_R)-x_b], \tag{3}$$

where $x(t)$ is the output intensity of the system, t_R is the delay time of the feedback loop, and A and x_b are the input intensity and bias voltage, respectively. The relaxation time of the system is 1. It is an infinite-dimensional system because an infinite set of independent numbers are required to specify an initial condition. For simplicity, we choose the initial condition $x(t)$ to be constant in the interval $[-t_R,0]$. This equation can be solved numerically, and a Runge-Kutta algorithm of fourth order is particularly suitable for that. Equation (3) is solved by keeping the parameters t_R and x_b fixed at 10 and 3, respectively. The number N of the phase space was fixed to 100. For a large enough N, the simulation gives the same results. At the end of this paper, we will give a numerical result for $N=200$ to verify it.

In order to eliminate the long-lived transients, we discarded the first 1×10^5 steps, and calculated the first ten Lyapunov exponents [5] for parameters from $A=2.0$ to 4.0. The Lyapunov spectrum diagram is shown in Fig. 1. As parameter A increases, the system goes to a chaotic state at $A\approx2.25$ through period-doubling cascades, and becomes a superchaos at $A\approx2.30$. There are two boundary crises, one at $A_{c1}=2.465\,948\,5$, and the other at $A_{c2}=3.699\,970\,5$, and the attractor in these two periodicity windows is frequency locked.

To determine the average transient lifetimes τ_1 and τ_2 for $A_1=2.47$ and $A_2=3.75$, we choose $N_0=3400$ initial conditions. Evolve these initial conditions under Eq. (3), and obtain the number of trajectories that have not settled into a frequency-locked attractor at time t. Figure 2 shows the number of chaotic trajectories $\ln(N(t))$ vs t in a semilogarithmic plot for $A_1=2.47$, where a trajectory is counted as chaotic at time t. The plot can be fitted by a straight line, indicating that the decay of a number of chaotic trajectories is exponential. The slope of the fitted line is $5.70\times10^{-5}\pm3.60\times10^{-7}$, which gives the average lifetime $\tau_1\approx1.80\times10^4\pm0.1\times10^4$. Using the same method for $A_2=3.75$, we obtain $\tau_2=1.30\times10^5$, a very long transient. In the window shown in Fig. 1, choosing a suitable initial condition, we find a very long chaotic transient motion, and calculate its Lyapunov diagram during the transient. Using the Kaplan-York conjecture [5], we can estimate the Lyapunov dimensions of the chaotic saddle to be 5.7 and 10.6 for A_1 and A_2, respectively.

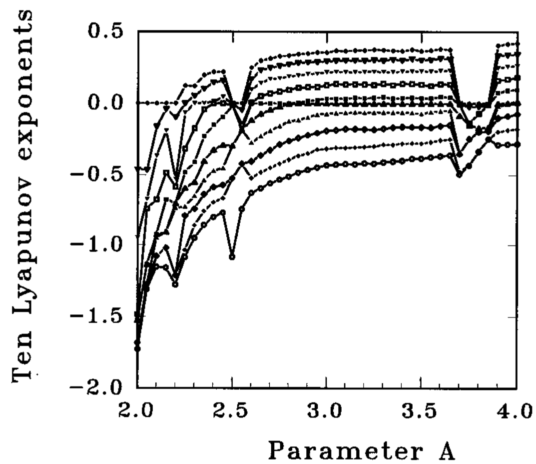

FIG. 1. First ten Lyapunov spectra for Eq. (3) from $A=2.0$–4.0 and fixed parameters $t_R=10$ and $x_b=3$. Two boundary crises take place at both $A_{c1}=2.465\,948\,5$ and $A_{c1}=3.699\,970\,5$.

Two attractors coexist in a system in which it is common for boundaries to exhibit a fractal structure [15]. From the practical point of view [14], we can suppose transient chaos to be a chaotic attractor within a large time t_c. Figure 3(a) shows the set of initial conditions drawn from 200 points in the region $1.0\leqslant x_0\leqslant2.0$ ($\varepsilon=5\times10^{-3}$) at $t_c=\tau_1(18\,000)$. C denotes initial conditions from which trajectories still wander chaotically during a long time t_c, and P denotes initial conditions from which trajectories settle into a final attractor during the time t_c. Figure 3(b) is blowup of Fig. 3(a) in the region $1.42\leqslant x_0\leqslant1.43$ ($\varepsilon=5\times10^{-5}$). Figure 3(c) is a blowup

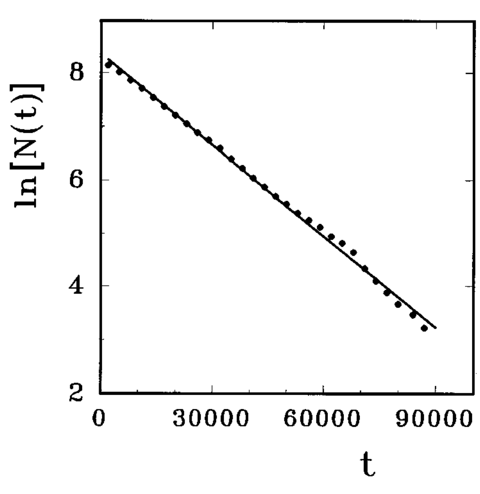

FIG. 2. The semilogarithmic plot of $N(t)$, the number of chaotic trajectories at time t. The average lifetime is $1.80\times10^4\pm0.10\times10^4$.

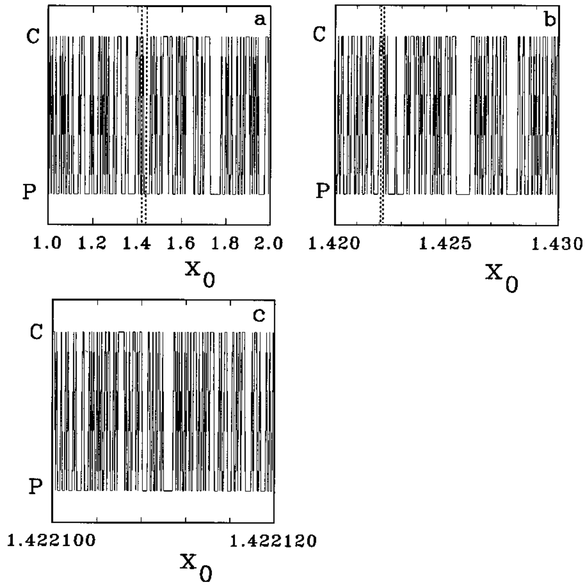

FIG. 3. The set of initial conditions drawn from 200 points at $t_c=\tau_1(1.80\times10^4)$ for $A=2.47$, where C denotes the initial conditions from which trajectories still wander chaotically during a long time t_c, and P denotes initial conditions from which trajectories settle into a final attractor during the time t_c. (a) For $1.0\leqslant x_0\leqslant 2.0$ ($\varepsilon=5\times10^{-3}$). (b) Blow up of the interval of $1.42\leqslant x_0\leqslant 1.43$ in (a) ($\varepsilon=5\times10^{-5}$). (c) Blowup of the interval of $1.4221\leqslant x_0\leqslant 1.42212$ in (b) ($\varepsilon=1\times10^{-7}$).

of Fig. 3(b) in the region $1.4221\leqslant x_0\leqslant 1.42212$ ($\varepsilon=10^{-7}$). We find that, they are similar in structure. For $t_c\approx 2.7\tau_1$(48 000), the results are shown in Fig. 4. Like Fig. 3, the parts of Fig. 4 have similar geometric structures. Figure 5 shows a similar result at $A=3.75$, taking t_c to be its

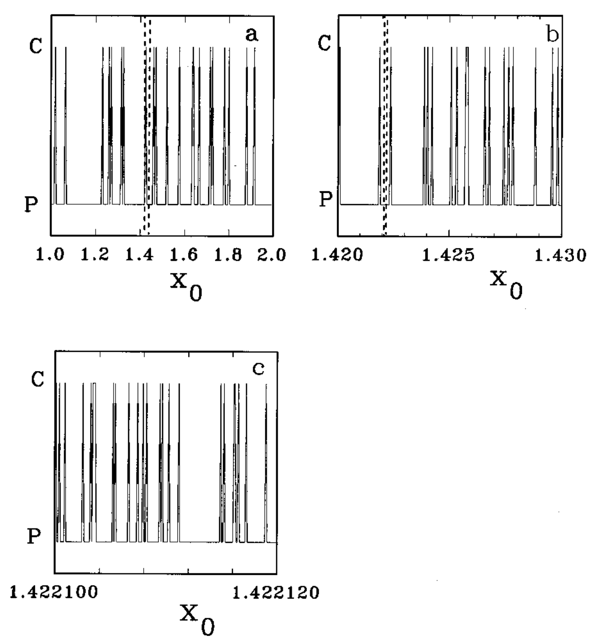

FIG. 4. Take $t_c\approx 2.7\tau_1$ (4.80×10^4), and the other parameters are the same as in Fig. 3.

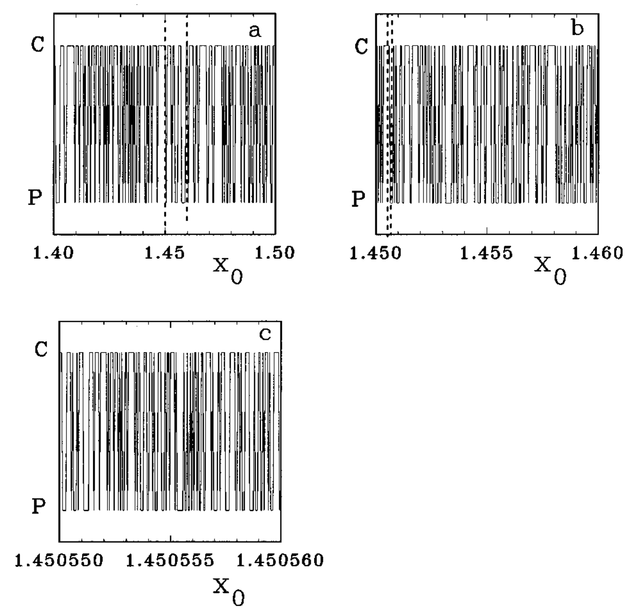

FIG. 5. The set of initial conditions drawn from 200 points at $t_c=\tau_2$ (1.30×10^4) for $A=3.75$. (a) For $1.40\leqslant x_0\leqslant 1.50$ ($\varepsilon=5\times10^{-3}$). (b) Blowup of the interval of $1.450\leqslant x_0\leqslant 1.460$ in (a) ($\varepsilon=5\times10^{-5}$). (c) Blowup of the interval of $1.45055\leqslant x_0\leqslant 1.45056$ in (b) ($\varepsilon=5\times10^{-8}$).

average lifetime τ_2. They indicate that arbitrarily long transient chaos still exists in any interval.

We have calculated the uncertainty exponent α. In order to do this, we computed a fraction of the uncertain initial condition $f(\varepsilon)$ for a given perturbation ε [15] by choosing 1000 initial conditions. Figure 6 shows $\log_{10} f(\varepsilon)$ vs $\log_{10}\varepsilon$ in a logarithmic plot. The uncertainty exponents are estimated to be $\alpha_1=0.0003\pm0.001$ for $A=2.47$ and $t_c=\tau_1$ [shown in Fig. 6(a)]. This indicates that the fraction of the uncertain initial condition is independent of ε. So we can obtain an average fraction of the uncertain initial condition f. In this case, $f_1=0.486\pm0.009$. Similarly, $\alpha_2=0.0000\pm0.0022$ for $A=3.75$ and $t_c=\tau_2$ [shown in Fig. 6(b)], and its average value $f_2=0.482\pm0.022$ which is consistent with f_1 within error. For $A=2.47$ and $t_c\approx 2.7\tau_1$, $\alpha_1'=0.0017\pm0.0024$ [shown in Fig. 6(c)], and its average value $f_1'=0.164\pm0.011$, which is smaller than f_1. The above results indicate that the uncertainty exponent α is independent of t_c.

To affirm that the above results are the same for enough large numbers of dimensions, we calculated τ_1 for $N=200$, The results are shown in Fig. 7. The average lifetime $\tau_1'=5.6\times10^{-5}\pm2.1\times10^{-7}$ [shown in Fig. 7(a)], so $\tau_1'\approx\tau_1$. We also calculated the uncertainty exponent $\alpha=0.0002\pm0.0013$ [shown in Fig. 7(b)]. This indicates that our results do not change with calculated dimension. For an infinite-dimensional system, phase space is infinite, and the stable manifold of the chaotic saddle is also infinite. It is meaningless to discuss the dimension of the manifold of an infinite system like a finite-dimensional system. However, we can still discuss the uncertainty exponent of the manifold. This uncertainty exponent is arbitrarily close to zero. This means that $f(\varepsilon)$ is independent of perturbation ε. In any scale and

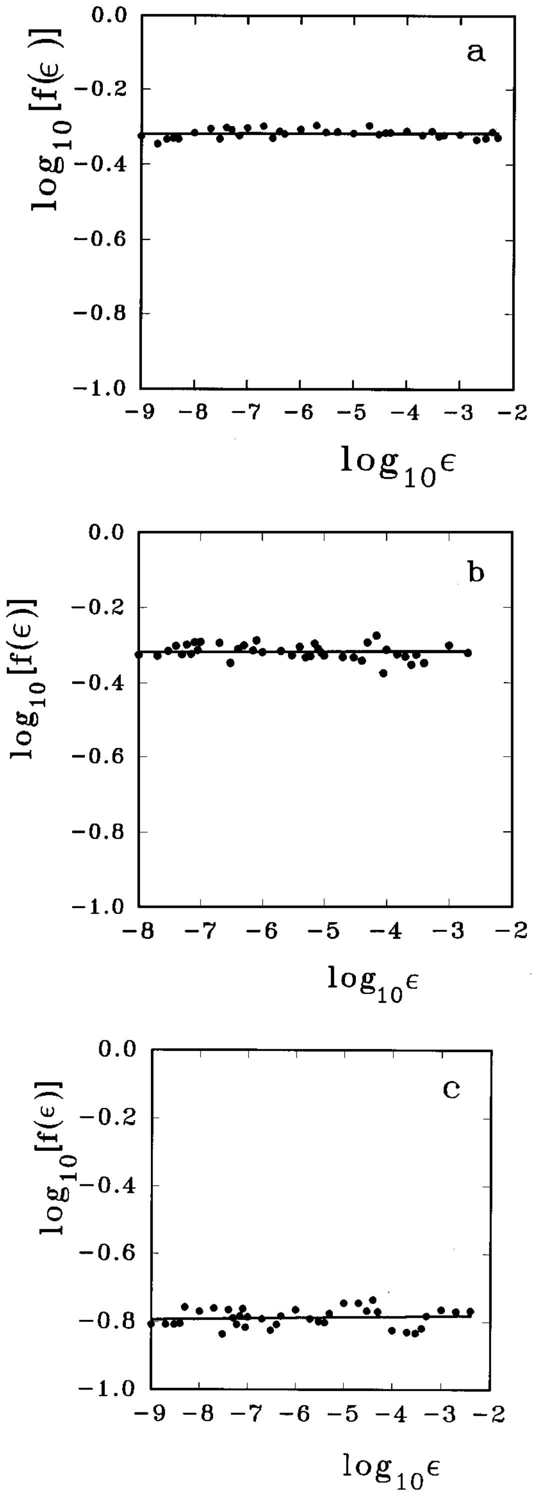

FIG. 6. Plot of the fraction of uncertain initial conditions $f(\varepsilon)$ vs the perturbation ε on a base-10 logarithmic scale. (a) The uncertainty exponent $\alpha_1=0.0003\pm0.001$ for $A=2.47$ and $t_c=\tau_1$. (b) $\alpha_2=0.0000\pm0.0022$ for $A=3.75$ and $t_c=\tau_2$. (c) $\alpha_1'=0.0017\pm0.0024$ for $A=2.47$ and $t_c\approx2.7\tau_1$.

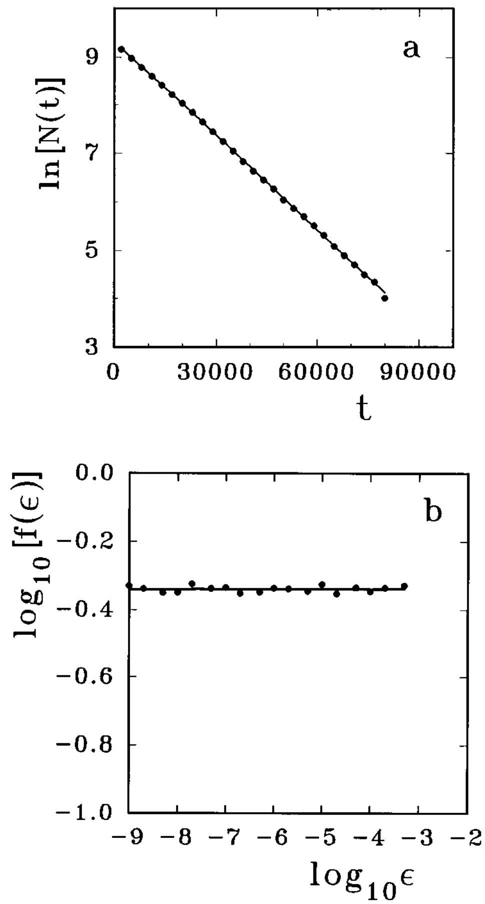

FIG. 7. Take the phase-space number of dimension $N=200$ and parameter $A=2.47$. (a) The average lifetime $\tau_1'=5.6\times10^{-5}\pm2.1\times10^{-7}$. (b) The uncertainty exponents $\alpha=0.0002\pm0.0013$.

any interval of initial conditions (except exactly on the chaotic saddle or its stable manifold, which measure zero), $f(\varepsilon)$ is a constant. Thus for a very long average lifetime which can occur in a chaotic saddle system with a very high Lyapunov dimension, we can find trajectories for the chaotic saddle or attractor which are random. This is different from the usual fractal basin boundary whose uncertainty exponent α is not zero. For this case, there is some interval of the initial condition in which any initial condition settles onto a certain attractor [16].

From the above results, our conclusions are as follows.

(a) Supertransients in delayed feedback optical bistable systems are due to an uncertainty exponent α arbitrarily close to zero, and we believe this conclusion is also correct for a class of delayed differential equations.

(b) The higher the Lyapunov dimension of a chaotic saddle, the longer its average lifetime, and the increase in the rates of the average lifetime is larger than that of its Lyapunov dimension. As the number of degrees of freedom tends to infinity, the average lifetime also tends to infinity, more quickly

(c) The fractions of the uncertain initial condition $f(\varepsilon)$ are the same if t_c is the average lifetime τ for different chaotic saddles with different Lyapunov dimensions.

(d) The uncertainty exponent α is independent of t_c.

This work was supported by the National Natural Science Foundation of China, and the Nonlinear Project of China.

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